

Functional analysis on two-dimensional local fields

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Abstract

We establish how a two-dimensional local field can be described as a locally convex space once an embedding of a local field into it has been fixed. We study the resulting spaces from a functional analytic point of view: in particular we characterize bounded, c-compact and compactoid submodules of two-dimensional local fields.

Introduction

This work is concerned with the study of characteristic zero two-dimensional local fields. These are complete discrete valuation fields whose residue field is a local field, either of characteristic zero or positive.

Following an idea introduced in [11], we do not regard two-dimensional fields as fields in the usual sense, but as an embedding of fields $K \hookrightarrow F$, where K is a local field and F is a two-dimensional local field. Given a two-dimensional local field F , such field embeddings always exist and we are not assuming any extra conditions on F ; we are only changing our point of view.

In the arithmetic-geometric context, such field embeddings arise in the following way (see [11] for details): suppose that S is the spectrum of the ring of integers of a number field and that $f : X \rightarrow S$ is an arithmetic surface (for our purposes it is enough to suppose that X is a regular 2-dimensional scheme and that f is projective and flat). Choose a closed point $x \in X$ and an irreducible curve $\{y\} \subset X$ passing through x . Suppose for simplicity that $x \in \{y\}$ is regular and let $s = f(x) \in S$. Starting from the local ring of regular functions $\mathcal{O}_{X,x}$, we obtain a two-dimensional local field $F_{x,y}$ through a process of repeated completions and localizations:

$$F_{x,y} = \text{Frac} \left(\widehat{\widehat{\mathcal{O}_{X,x,y}}} \right).$$

This is analogue to the procedure of completion and localization that allows us to obtain a local field $K_s = \text{Frac}(\widehat{\mathcal{O}_{S,s}})$. The structure morphism $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ induces a field embedding $K_s \hookrightarrow F_{x,y}$.

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The moral of the above paragraph is that if two-dimensional fields arise from an arithmetic-geometric context then they always come with a prefixed local field embedded into them.

What we study in this work is the K -vector space structure associated to F via the embedding $K \hookrightarrow F$. As such, we connect the topological theory of two-dimensional local fields with the theory of nonarchimedean locally convex vector spaces. In particular, for the fields $K((t))$ and $K\{\{t\}\}$ (see §2 for the definition of the latter), we establish in Corollaries 3.2 and 3.7, a family of defining seminorms for the higher topology of the form

$$\left\| \sum_i x_i t^i \right\| = \sup_i |x_i| q^{n_i},$$

where $\{n_i\}_{i \in \mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$ is a sequence subject to certain conditions and q is the number of elements in the residue field of K .

In particular, this provides us with a new way to describe higher topologies on two-dimensional local fields which does not rely on taking a lifting map from the residue field as in [9]. This also introduces a new concept of bounded subset.

Although our description of higher topologies is valid for both equal and mixed characteristic two-dimensional local fields, the study of the functional theoretic properties in the two cases suggests that similarities stop here. Equal characteristic fields may be shown to be LF-spaces (a direct limit of Fréchet spaces) and, as such, they are bornological, nuclear and reflexive. This characterization is unavailable for mixed characteristic fields such as $K\{\{t\}\}$ and we show how these properties do not hold.

One of the advantages of our point of view is that certain submodules of F arise as the families of c-compact and compactoid submodules, and therefore have a property which is a linear-topological analogue of compactness. In particular, compactoid submodules coincide with bounded submodules in equal characteristic (this is a consequence of nuclearity but can be made very explicit, cf. Proposition 5.8) and define a family strictly contained in that of bounded submodules in mixed characteristic. By using the associated bornology we achieve in Theorem 6.3 a very explicit self-duality result which is hinted at in [3, §3].

We briefly outline the contents of this work. Sections §1 and §2 summarise certain bits of the theory of nonarchimedean locally convex vector spaces and the structure of two-dimensional local fields, respectively. We have included them in this work in order to be able to refer to certain general results in later parts of the work and in order to fix notations and conventions. Hence, we do not supply proof for any statement in these sections, but refer the reader to the available literature.

In section §3 we describe how higher topologies induce the structure of locally convex K -vector spaces on $K((t))$ and $K\{\{t\}\}$. The main results are Propositions 3.1 and 3.6, along with Corollaries 3.2 and 3.7, which describe higher topologies in terms of seminorms.

Sections §4, §5, §6 and §7 develop a systematic study of the locally convex vector spaces introduced in §3. Those sections study bounded sets and bornolog-

ical properties; complete, c-compact and compactoid submodules; duality and nuclearity, respectively.

In §8, we extend the results of the previous sections to the case of a general embedding $K \hookrightarrow F$ of a local field into a two-dimensional local field.

Sections §9 and §10 explain how the results in this work can also be applied to archimedean two-dimensional local fields and positive characteristic local fields, respectively. In the first case, we are dealing with LF-spaces and we deduce our results from the well-established theory of (archimedean) locally convex spaces. In the second case, we relate the locally convex structure of vector spaces over $\mathbb{F}_q((u))$ to the linear topological structure of vector spaces over \mathbb{F}_q through restriction of scalars. In this case we recover already known results of Beilinson [1] and Kapranov [6].

Finally, we discuss some applications and further directions of research in §11.

Notation. Whenever F is a complete discrete valuation field, we will denote by $\mathcal{O}_F, \mathfrak{p}_F, \pi_F$ and \overline{F} its ring of integers, the unique prime ideal in the ring of integers, an element of valuation one and the residue field, respectively. A two-dimensional local field is a complete discrete valuation field F such that \overline{F} is a local field.

Throughout the text, K will denote a characteristic zero local field, that is, a finite extension of \mathbb{Q}_p for some prime p . The cardinality of the finite field \overline{K} will be denoted by q . The absolute value of K will be denoted by $|\cdot|$, normalised so that $|\pi_K| = q^{-1}$. Due to far too many appearances in the text, we will ease notation by letting $\mathcal{O} := \mathcal{O}_K$, $\mathfrak{p} := \mathfrak{p}_K$ and $\pi := \pi_K$.

The conventions $\mathfrak{p}^{-\infty} = K$, $\mathfrak{p}^{\infty} = \{0\}$ and $q^{-\infty} = 0$ will be used.

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1 Locally convex spaces over K

In this section we summarise some concepts and fix some notation regarding locally convex vector spaces over K . This is both for the reader's convenience as much as for establishing certain statements and properties for later reference.

The theory of locally convex vector spaces over a nonarchimedean field is well developed in the literature, so we will keep a concise exposition of the facts that we will require later. Details and proofs for results stated in this section may be found in [13]; we often use similar notations.

Let V be a K -vector space. A lattice in V is an \mathcal{O} -submodule $\Lambda \subseteq V$ such that for any $v \in V$ there is an element $a \in K^\times$ such that $av \in \Lambda$. This is equivalent to having

$$\Lambda \otimes_{\mathcal{O}} K \simeq V$$

as K -vector spaces. A subset of V is said to be convex if it is of the form $v + \Lambda$ for $v \in V$ and Λ a lattice in V . A vector space topology on V is said to be locally convex if the filter of neighbourhoods of zero admits a collection of lattices as a basis.

A seminorm on V is a map $\|\cdot\| : V \rightarrow \mathbb{R}$ such that:

- (i) $\|\lambda v\| = |\lambda| \cdot \|v\|$ for every $\lambda \in K$, $v \in V$,
- (ii) $\|v + w\| \leq \max(\|v\|, \|w\|)$ for all $v, w \in V$.

These conditions imply in particular that a seminorm only takes non-negative values and that $\|0\| = 0$.

The gauge seminorm of a lattice $\Lambda \subseteq V$ is defined by the rule:

$$\|\cdot\|_\Lambda : V \rightarrow \mathbb{R}, \quad v \mapsto \inf_{v \in a\Lambda} |a|. \quad (1)$$

Given a family of seminorms $\{\|\cdot\|_j\}_{j \in J}$ on V , there is a unique coarsest vector space topology on V making the maps $\|\cdot\|_j : V \rightarrow \mathbb{R}$ continuous for every $j \in J$. Such topology is locally convex, as the open balls centered at zero for the defining seminorms supply a basis of neighbourhoods of zero consisting of open lattices.

A locally convex topology can be described in terms of lattices or in terms of seminorms; passing from one point of view to the other is a simple matter of language.

A subset $B \subset V$ is bounded if for any open lattice $\Lambda \subset V$, there is an $a \in K$ such that $B \subseteq a\Lambda$. Alternatively, B is bounded if for every continuous seminorm $\|\cdot\|$ on V we have

$$\sup_{v \in B} \|v\| < \infty.$$

A locally convex K -vector space V is bornological if any seminorm which is bounded on bounded sets is continuous.

The collection of bounded sets of V defines a bornology, that is: a collection of subsets of V which is stable under finite unions and hereditary by inclusion [5]. Just like a topology on a set is the minimum amount of information required in order to have a notion of open set and continuous map, a bornology on a set is the minimum amount of information required in order to have a notion of bounded set and bounded map.

Bornological locally convex spaces are the ones for which topology is determined by bornology: a linear map from such a space into another space is continuous if and only if it is bounded.

On an arbitrary vector space, if a convex bornology is specified, we get a bornological locally convex space by considering the strongest locally convex topology that gives rise to the specified family of bounded sets.

Open lattices in a non-archimedean locally convex space are also closed. The space V is said to be barrelled if any closed lattice is open.

Among many general ways to construct locally convex spaces [13, §5], we will require the use of products.

Proposition 1.1. *Let $\{V_i\}_{i \in I}$ be a family of locally convex K -vector spaces, and let $V = \prod_{i \in I} V_i$. Then the product topology on V is locally convex.*

If $\{\Lambda_{i,j}\}_j$ denotes the set of open lattices of V_i for $i \in I$, then the set of open lattices of V is given by finite intersections of lattices of the form $\pi_i^{-1}\Lambda_{i,j}$.

Equivalently, the product topology on V is the one defined by all seminorms of the form

$$v \mapsto \sup_{i,j} \|\pi_i(v)\|_{i,j},$$

where $\{\|\cdot\|_{i,j}\}_j$ is a defining family of seminorms for V_i for all $i \in I$, $\pi_i : V \rightarrow V_i$ is the corresponding projection and the supremum is taken over a finite collection of indices i, j .

Another construction which we will require is that of inductive limits. Let V be a K -vector space and $\{V_i\}_{i \in I}$ be a collection of locally convex K -vector spaces. Let, for each $i \in I$, $f_i : V_i \rightarrow V$ be a K -linear map. The final topology for the collection $\{f_i\}_{i \in I}$ is not locally convex in general. However, there is a finest locally convex topology on V making the map f_i continuous for every $i \in I$. That topology is called the *locally convex final* topology on V . Inductive limits and direct sums of locally convex spaces are particular examples of such construction.

Definition 1.2. Suppose that V is a K -vector space and that we have an increasing sequence of vector subspaces $V_1 \subseteq V_2 \subseteq \dots \subseteq V$ such that $V = \bigcup_{n \in \mathbb{N}} V_n$. Suppose that for each $n \in \mathbb{N}$, V_n is equipped with a locally convex topology such that $V_n \hookrightarrow V_{n+1}$ is continuous. Then the final locally convex topology on V is called the strict inductive limit topology

Next, we need to discuss completeness issues. We require to deal not only with sequences, but arbitrary nets. Let I be a directed set and V a locally convex K -vector space. A net in V is a family of vectors $(v_i)_{i \in I} \subset V$. A sequence is a net which is indexed by the set of natural numbers.

The net $(v_i)_{i \in I}$ converges to a vector v , and we shall write $v_i \rightarrow v$, if for any $\varepsilon > 0$ and continuous seminorm $\|\cdot\|$ on V , there is an index $i \in I$ such that for every $j \geq i$ we have $\|v_j - v\| \leq \varepsilon$.

Similarly, the net $(v_i)_{i \in I}$ is said to be Cauchy if for any $\varepsilon > 0$ and continuous seminorm $\|\cdot\|$ on V there is an index $i \in I$ such that for every pair of indices $j, k \geq i$ we have $\|v_j - v_k\| \leq \varepsilon$.

Definition 1.3. A subset $A \subseteq V$ is said to be complete if any Cauchy net in A converges to a vector in A .

Example 1.4. K is complete for nets, since it is a normed K -vector space.

The usual topological notion of compactness is not very powerful for the study of infinite dimensional vector spaces over non-archimedean fields. In our case, there is an \mathcal{O} -linear concept of compactness which is a good substitute for compactness.

Definition 1.5. Let A be an \mathcal{O} -submodule of V . A is said to be c -compact if, for any decreasingly filtered family $\{\Lambda_i\}_{i \in I}$ of open lattices of V , the canonical map

$$A \rightarrow \varprojlim_{i \in I} A/(\Lambda_i \cap A)$$

is surjective.

Example 1.6. The base field K is c-compact as a K -vector space. This shows that a c-compact module need not be bounded.

This property may be phrased in a more topological way.

Proposition 1.7. *An \mathcal{O} -submodule $A \subseteq V$ is c-compact if and only if for any family $\{C_i\}_{i \in I}$ of closed convex subsets $C_i \subseteq A$ such that $\bigcap_{i \in I} C_i = \emptyset$ there are finitely many indices $i_1, \dots, i_m \in I$ such that $C_{i_1} \cap \dots \cap C_{i_m} = \emptyset$.*

Proof. See [13, Lemma 12.1.ii and subsequent paragraph]. \square

Proposition 1.8. *Let $\{V_h\}_{h \in H}$ be a collection of locally convex K -vector spaces, and for each $h \in H$ let $A_h \subseteq V_h$ be a c-compact \mathcal{O} -submodule. Then $\prod_{h \in H} A_h$ is c-compact in $\prod_{h \in H} V_h$.*

Proof. [13, Prop. 12.2]. \square

Another notion which is used in this setting is that of a compactoid \mathcal{O} -module; it is a notion which is analogous to that of *relative compactness*.

Definition 1.9. Let $A \subseteq V$ be an \mathcal{O} -submodule. A is compactoid if for any open lattice Λ of V there are finitely many vectors $v_1, \dots, v_m \in V$ such that

$$A \subseteq \Lambda + \mathcal{O}v_1 + \dots + \mathcal{O}v_m.$$

Let $A \subseteq V$ be an \mathcal{O} -submodule. If A is c-compact, then it is closed and complete. Similarly, if A is compactoid then it is bounded. [13, §12].

Proposition 1.10. *Let $A \subseteq V$ be an \mathcal{O} -submodule. The following are equivalent.*

- (i) A is c-compact and bounded.
- (ii) A is compactoid and complete.

Proof. [13, Prop. 12.7]. \square

The collection of compactoid \mathcal{O} -submodules of V generates a bornology which is a priori weaker than the one given by the locally convex topology.

If V, W are two locally convex K -vector spaces, a linear map $f : V \rightarrow W$ is continuous as soon as the pull-back of a continuous seminorm is a continuous seminorm. We denote the K -vector space of continuous linear maps between V and W by $\mathcal{L}(V, W)$.

The space $\mathcal{L}(V, W)$ may be topologized in the following way. Let \mathcal{B} be a collection of bounded subsets of V . For any continuous seminorm $\|\cdot\|$ on W and $B \in \mathcal{B}$, consider the seminorm

$$\|\cdot\|_B : \mathcal{L}(V, W) \rightarrow \mathbb{R}, \quad f \mapsto \sup_{v \in B} \|f(v)\|.$$

Definition 1.11. We write $\mathcal{L}_{\mathcal{B}}(V, W)$ for the space of continuous linear maps from V to W endowed with the locally convex topology defined by the seminorms $\|\cdot\|_B$, for every continuous seminorm $\|\cdot\|$ on W and $B \in \mathcal{B}$.

In the particular case in which \mathcal{B} consists of all bounded sets of V , we write $\mathcal{L}_b(V, W)$ for the resulting space, which is then said to have the topology

of uniform convergence. If \mathcal{B} consists only of the singletons $\{v\}$ for $v \in V$, we denote the resulting space by $\mathcal{L}_s(V, W)$ and say that it has the topology of point-wise convergence. Finally, if \mathcal{B} is the collection of compactoid \mathcal{O} -submodules of V , we denote the resulting space by $\mathcal{L}_c(V, W)$.

There are two cases of particular interest: the topological dual space $V' = \mathcal{L}(V, K)$, and the endomorphism ring $\mathcal{L}(V) = \mathcal{L}(V, V)$. We denote $F'_s, F'_b, F'_c, \mathcal{L}_s(V), \mathcal{L}_b(V)$ and $\mathcal{L}_c(V)$ for the corresponding topologies of point-wise convergence, uniform convergence and uniform convergence on compactoid submodules, respectively.

The notion of polarity plays a role in the study of duality, as it provides us with a way of relating \mathcal{O} -submodules of V to \mathcal{O} -submodules of V' .

Definition 1.12. If $A \subseteq V$ is an \mathcal{O} -submodule, we define its pseudopolar by

$$A^p = \{l \in V'; |l(v)| < 1 \text{ for all } v \in A\}.$$

The pseudobipolar of A is

$$A^{pp} = \{v \in V; |l(v)| < 1 \text{ for all } l \in A^p\}.$$

We have that, $l \in A^p$ if and only if $l(A) \subseteq \mathfrak{p}$. Note that the traditional notion of polar relaxes the condition in the definition of pseudopolar to $|l(v)| \leq 1$ or, equivalently, $l(A) \subseteq \mathcal{O}$. Introducing the distinction is an important technical detail, as pseudo-polarity is a better-behaved notion in the nonarchimedean setting.

Pseudopolarity provides with a way to relate \mathcal{O} -submodules of V to \mathcal{O} -submodules in V' .

Proposition 1.13. *Let $A \subseteq V$ be an \mathcal{O} -submodule. We have*

- (i) *If $A \subseteq B \subseteq V$ is another \mathcal{O} -submodule, then $B^p \subseteq A^p$.*
- (ii) *A^p is closed in V'_s .*
- (iii) *If $A \in \mathcal{B}$, then A^p is an open lattice in V'_B .*

Proof. This is part of [13, Lemma 13.1]. □

In order to conclude this section we define nuclear spaces. For any submodule $A \subseteq V$, denote $V_A := A \otimes_{\mathcal{O}} K$, endowed with the locally convex topology associated to the gauge seminorm $\|\cdot\|_A$. V_A may well not be a Hausdorff space, but its completion

$$\widehat{V}_A := \varprojlim_{n \in \mathbb{Z}} V_A / \pi^n A$$

is a K -Banach space.

Definition 1.14. V is said to be nuclear if for any open lattice $\Lambda \subseteq V$ there exists another open lattice $M \subseteq \Lambda$ such that the canonical map $\widehat{V}_M \rightarrow \widehat{V}_\Lambda$ is compact, that is: there is an open lattice in \widehat{V}_M such that the closure of its image is bounded and c-compact.

Proposition 1.15. *We have:*

(i) An \mathcal{O} -submodule of a nuclear space is bounded if and only if it is compactoid.

(ii) Arbitrary products of nuclear spaces are nuclear.

(iii) Strict inductive limits of nuclear spaces are nuclear.

Proof. (i) is [13, Proposition 19.2], (ii) is [13, Proposition 19.7] and (iii) is [13, Corollary 19.8]. \square

2 Our point of view on two-dimensional local fields

We consider the category whose objects are field inclusions

$$K \hookrightarrow F$$

where K is our fixed characteristic zero local field and F is a two-dimensional local field. In such case, we shall say that F is a two-dimensional local field over K . A morphism in this category between $K \hookrightarrow F_1$ and $K \hookrightarrow F_2$ is a commutative diagram of field inclusions

$$\begin{array}{ccc} F_1 & \longrightarrow & F_2 \\ \uparrow & \nearrow & \\ K & & \end{array}$$

where $F_1 \hookrightarrow F_2$ is an extension of complete discrete valuation fields.

The classification of two-dimensional local fields follows from Cohen structure theory of complete local rings and was established in [9]. The particular case with which we are dealing is very well described in [11, §2.2 and 2.3].

By this classification, given a two-dimensional local field F it is always possible to exhibit a local field contained in it, so our assumption does not imply any further structure on F . Let us briefly recall the structure of two-dimensional local fields, which depends on the relation between the characteristics of F and \overline{F} .

If $\text{char } F = \text{char } \overline{F}$, the choice of a uniformizer t for the discrete valuation of F determines an isomorphism $F \simeq \overline{F}((t))$. Such an isomorphism is not unique, as it depends on the choice of a coefficient field $\overline{F} \hookrightarrow F$.

Besides fields of Laurent series, there is another construction which is key in order to work with two-dimensional local fields, and higher local fields in general. For any complete discrete valuation field L , consider

$$L_{\{\{t\}\}} = \left\{ \sum_{i \in \mathbb{Z}} x_i t^i; x_i \in L, \inf v_L(x_i) > -\infty, a_i \rightarrow 0 (i \rightarrow -\infty) \right\},$$

with operations given by the usual addition and multiplication of power series. Note that we need to use convergence of series in L in order to define the product. With the discrete valuation given by

$$v_{L_{\{\{t\}\}}} \left(\sum_{i=-\infty}^{\infty} a_i t^i \right) := \inf v_L(a_i),$$

$L\{\{t\}\}$ turns into a complete discrete valuation field and the extension $L\{\{t\}\} | L$ is an unramified extension of complete discrete valuation fields.

When L is a characteristic zero local field, the field $L\{\{t\}\}$ turns into a 2-dimensional local field which we call the standard mixed characteristic field over L . Its first residue field is $\overline{L}((\overline{t}))$.

We view elements of L as elements of $L\{\{t\}\}$ in the obvious way. In particular, if π_L is a uniformizer of \mathcal{O}_L , it is also a uniformizer of $\mathcal{O}_{L\{\{t\}\}}$; the element $t \in L\{\{t\}\}$ is such that $\overline{t} \in \overline{L}((\overline{t}))$ is a uniformizer.

Suppose now that F is any two-dimensional local field such that $\text{char } F \neq \text{char } \overline{F}$, and that an embedding $K \hookrightarrow F$ is given. In this case, F contains a subfield which is K -isomorphic to $K\{\{t\}\}$, and this extension is finite.

Regardless of $\text{char } \overline{F}$, the existence of a field inclusion $K \subset F$ forces a certain compatibility between the rank-two valuation of F and the discrete valuation of K . Namely, we have an inclusion of abelian groups $K^\times \subset F^\times$. The structure of these abelian groups is well known, and implies that one of the components of the rank-two valuation of F restricts to the discrete valuation of K and the other one restricts trivially.

Example 2.1. Consider $K = \mathbb{Q}_p \subset \mathbb{Q}_p\{\{t\}\} = F$. In such case, the rank-two valuation of F is

$$(v_1, v_2) : F^\times \rightarrow \mathbb{Z} \oplus \mathbb{Z}, \quad \sum_{i \in \mathbb{Z}} a_i t^i \mapsto \left(\inf_{i \in \mathbb{Z}} v_p(a_i), \inf \{i; a_i \notin p\mathbb{Z}_p\} \right).$$

The restriction of v_1 to K is v_p , while v_2 restricts trivially.

Example 2.2. Consider $K = \mathbb{Q}_p \subset \mathbb{Q}_p((t)) = F$. In such case, the rank-two valuation of F is

$$(v_1, v_2) : F^\times \rightarrow \mathbb{Z} \oplus \mathbb{Z}, \quad \sum_{i \geq i_0} a_i t^i \mapsto (i_0, v_p(a_{i_0})),$$

where we suppose that a_{i_0} is the first nonzero coefficient in the power series. The restriction of v_1 to K is trivial while the restriction of v_2 to K is v_p .

Remark 2.3. There are two particular local fields which play a very distinguished role when these objects are to be studied from a functional analytic point of view. Those are the fields $K((t))$ and $K\{\{t\}\}$. As we will see, most topological properties which hold in these particular cases will hold in general after taking restrictions of scalars or a base change over a finite extension which topologically is equivalent to taking a finite cartesian product. It is for this reason that we will work from now on with these two particular examples of two-dimensional local fields. We will explain how our results extend to the general case in §8.

Notation. When working with the two-dimensional local fields $F = K\{\{t\}\}$ or $F = K((t))$, for any collection $\{A_i\}_{i \in \mathbb{Z}}$ of subsets of K , we will denote

$$\sum_{i \in \mathbb{Z}} A_i t^i = \left\{ \sum_i x_i t^i \in F; x_i \in A_i \text{ for all } i \in \mathbb{Z} \right\}.$$

We will also denote $\mathcal{O}_{K\{\{t\}\}} = \mathcal{O}\{\{t\}\}$. After all, this ring consists of all power series in $K\{\{t\}\}$ all of whose coefficients lie in \mathcal{O} .

3 Higher topologies are locally convex

In this section we will explain how the higher topology on $K((t))$ and $K\{\{t\}\}$ is a locally convex topology.

We are forced to study both cases separately.

3.1 Equal characteristic

The higher topology on $K((t))$ is defined as follows. Let $\{U_i\}_{i \in \mathbb{Z}}$ be a collection of open neighbourhoods of zero in K such that, if i is large enough, $U_i = K$. Then define

$$\mathcal{U} = \sum_{i \in \mathbb{Z}} U_i t^i. \quad (2)$$

The collection of sets of the form \mathcal{U} defines the set of neighbourhoods of zero of a group topology on $K((t))$.

Proposition 3.1. *The higher topology on $K((t))$ defines the structure of a locally convex K -vector space.*

Proof. As K is a local field, the collection of open neighbourhoods of zero admits a collection of open subgroups as a filter, that is: the basis of neighbourhoods of zero for the topology is generated by the sets of the form

$$\mathfrak{p}^n = \{a \in K; v_K(a) \geq n\},$$

where $n \in \mathbb{Z} \cup \{-\infty\}$. These closed balls are not only subgroups, but \mathcal{O} -fractional ideals. This in particular means that the sets of the form

$$\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i \subseteq K((t)), \quad (3)$$

where $n_i = -\infty$ for large enough i , generate the higher topology. Moreover, they are not only additive subgroups, but also \mathcal{O} -modules.

If $x = \sum_{i \geq i_0} x_i t^i \in K((t))$ is an arbitrary element, and i_1 is such that $n_i = -\infty$ for all $i > i_1$ then we have the possibilities:

(i) $i_0 < i_1$, in which case $x \in \Lambda$.

(ii) $i_1 \leq i_0$. In such case, let

$$n = \max \left(\max_{i_0 \leq i \leq i_1} n_i, 0 \right).$$

Then $\pi^n \in \mathcal{O}$ satisfies $\pi^n x \in \Lambda$.

Thus, Λ is a lattice and the higher topology is locally convex. \square

As a consequence of the previous proposition, it is possible to describe the higher topology in terms of seminorms.

Corollary 3.2. *For any sequence $(n_i)_{i \in \mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$ such that there is an integer k satisfying $n_i = -\infty$ for all $i > k$, define*

$$\|\cdot\| : K((t)) \rightarrow \mathbb{R}, \quad \sum_{i \gg -\infty} x_i t^i \mapsto \max_{i \leq k} |x_i| q^{n_i}. \quad (4)$$

Then $\|\cdot\|$ is a seminorm on $K((t))$ and the higher topology on $K((t))$ is the locally convex topology defined by the family of seminorms given by (4) as $(n_i)_{i \in \mathbb{Z}}$ describes all sequences satisfying the above restrictions.

Proof. This result is a consequence of Proposition 3.1 and of the fact that the gauge seminorm attached to a lattice of the form

$$\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i$$

with $n_i = \infty$ for all $i > k$ is precisely the one given by (4). In order to see that, let $x = \sum_{i \geq i_0} x_i t^i \in K((t))$ and $a \in K$. We have that $x \in a\Lambda$ if and only if $x_i \in a\mathfrak{p}^{n_i}$ for every $i \geq i_0$. This is the case if and only if we have

$$|x_i|q^{n_i} \leq |a|$$

for all $i \geq i_0$. The infimum value of $|a|$ for which the above inequality holds is precisely the supremum of the values of $|x|q^{n_i}$ for $i \geq i_0$. \square

The seminorm $\|\cdot\|$ from the previous corollary is associated to and does depend on the choice of the sequence $(n_i)_{i \in \mathbb{Z}}$. If we have chosen notation not to reflect this fact, it is in hope that a lighter notation will simplify reading and that the sequence of integers defining $\|\cdot\|$, when needed, will be clear from the context.

Remark 3.3. As F is a field, it is worth asking ourselves whether the seminorm (4) is multiplicative. It is very easy to check that for $i, j \in \mathbb{Z}$,

$$\|t^i\| \cdot \|t^j\| = q^{n_i + n_j},$$

while

$$\|t^{i+j}\| = q^{n_{i+j}}.$$

These two values need not coincide in general.

The fact below is not new, but we state it here for our convenience.

Proposition 3.4. *The subspace topology induced on $K[[t]]$ agrees with the product topology on $K^{\mathbb{N}}$.*

Proof. The open lattices for the product topology on $K^{\mathbb{N}}$ are exactly the ones of the form

$$\prod_{i \in I} \Lambda_i \times \prod_{i \notin I} K,$$

where I is a finite subset of \mathbb{N} and Λ_i are open lattices in K , that is, integer powers of \mathfrak{p} . This description agrees with the description of the open lattices in $K[[t]]$ for the subspace topology. \square

Remark 3.5. After the previous proposition, we can describe the higher topology on $K((t))$ as follows: consider the product topology on $t^{-i}K[[t]] \simeq K^{\mathbb{N}}$ and topologize

$$K((t)) = \cup_{i \in \mathbb{N}} t^{-i}K[[t]]$$

using the strict inductive limit topology (Definition 1.2). This in particular implies that the higher topology on $K((t))$ may be described by taking simpler seminorms. The present formulation will be useful to us as it is common for both $K((t))$ and $K\{\{t\}\}$.

3.2 Mixed characteristic

The higher topology on $K\{\{t\}\}$ may be described as follows.

Let $\{V_i\}_{i \in \mathbb{Z}}$ be a sequence of open neighbourhoods of zero in K such that

- (i) There is $c \in \mathbb{Z}$ such that $\mathfrak{p}^c \subseteq V_i$ for every $i \in \mathbb{Z}$.
- (ii) For every $l \in \mathbb{Z}$ there is an index $i_0 \in \mathbb{Z}$ such that $\mathfrak{p}^l \subseteq V_i$ for every $i \geq i_0$.

Then define

$$\mathcal{V} = \sum_{i \in \mathbb{Z}} V_i t^i \subset K\{\{t\}\}. \quad (5)$$

The higher topology on $K\{\{t\}\}$ is the group topology defined by taking the sets of the form \mathcal{V} as the collection of open neighbourhoods of zero [9, §1].

Again, as K is a local field, the collection of neighbourhoods of zero admits the collection of open subgroups as a filter. These are not only subgroups but \mathcal{O} -fractional ideals, namely the integer powers of the prime ideal \mathfrak{p} .

Let $(n_i)_{i \in \mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$, restricted to the conditions:

- (i) There is $c \in \mathbb{Z}$ such that $n_i \leq c$ for every i .
- (ii) For every $l \in \mathbb{Z}$ there is an index $i_0 \in \mathbb{Z}$ such that $n_i \leq l$ for every $i \geq i_0$.

Condition (ii) is equivalent, by definition of limit of a sequence, to having $n_i \rightarrow -\infty$ as $i \rightarrow \infty$; we will phrase it this way in the future.

Proposition 3.6. *The set*

$$\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i \quad (6)$$

is an \mathcal{O} -lattice. The sets of the form (6) generate the higher topology on $K\{\{t\}\}$, which is locally convex.

Proof. It is clear that Λ is an \mathcal{O} -module, and that the conditions imposed on the indices n_i imply that it is a basic neighbourhood of zero for the higher topology.

Given an arbitrary element $x = \sum_{i=-\infty}^{\infty} x_i t^i \in F$, we must show the existence of an element $a \in K^\times$ such that $ax \in \Lambda$. Indeed, a power of the uniformizer does the trick: we have that $\pi^n x \in \Lambda$ if and only if $\pi^n x_i \in \mathfrak{p}^{n_i}$ for every $i \in \mathbb{Z}$, and this is true if and only if

$$n + v_K(x_i) \geq n_i$$

for all $i \in \mathbb{Z}$. In other words, such an n exists if and only if the difference

$$n_i - v_K(x_i)$$

cannot be arbitrarily large. But on one hand there is an integer c that bounds the n_i from above, and on the other hand the values $v_K(x_i)$ are bounded below by $v_F(x)$. We may take $n = c - v_F(x)$.

Because the integer powers of \mathfrak{p} generate the basis of neighbourhoods of zero of the higher topology on F , the lattices of the form (6) generate the higher topology. In particular, the higher topology on $K\{\{t\}\}$ is locally convex. \square

Once we know that the higher topology is locally convex, we can describe it in terms of seminorms.

Corollary 3.7. *Let $(n_i)_{i \in \mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$ be a sequence subject to the conditions:*

- (i) *There is $c \in \mathbb{Z}$ such that $n_i \leq c$ for all $i \in \mathbb{Z}$.*
- (ii) *We have $n_i \rightarrow -\infty$ as $i \rightarrow \infty$.*

Consider the seminorm

$$\|\cdot\| : K\{\{t\}\} \rightarrow \mathbb{R}, \quad \sum_{i \in \mathbb{Z}} x_i t^i \mapsto \sup_{i \in \mathbb{Z}} |x_i| q^{n_i}. \quad (7)$$

Then, the higher topology on $K\{\{t\}\}$ is the locally convex topology generated by the family of seminorms defined by (7), as $(n_i)_{i \in \mathbb{Z}}$ ranges through the sequences specified above.

Proof. The gauge seminorm associated to the lattice Λ is (7). The argument is the same as the proof of Corollary 3.2 and we omit it. \square

The seminorms in Corollary 3.7 are well defined because they arise as gauge seminorms attached to lattices. If we forget this fact for a moment, let us examine the values $|x_i| q^{n_i}$.

On one hand, when i tends to $-\infty$, the values $|x_i|$ tend to zero while the values q^{n_i} stay bounded. On the other hand, when i tends to $+\infty$ the values $|x_i|$ stay bounded and q^{n_i} tends to zero. In conclusion, the values $|x_i| q^{n_i}$ are all positive and tend to zero when $|i| \rightarrow +\infty$; this implies the existence of their supremum.

Just like in the equal characteristic case, a defining seminorm $\|\cdot\|$ is not multiplicative, for the same reason.

A mixed characteristic two-dimensional local field cannot be viewed as a direct limit in a category of locally convex K -vector spaces in the fashion of Remark 3.5. However, such an approach is valid from an algebraic point of view in a category of \mathcal{O} -modules.

3.3 First topological properties

For starters, let us recall from [9] a few well-known properties of higher topologies. A two-dimensional local field $K \hookrightarrow F$ endowed with a higher topology is a Hausdorff topological group. Moreover, multiplication by a fixed nonzero element defines a homeomorphism $F \rightarrow F$ and the residue map $\mathcal{O}_F \rightarrow \overline{F}$ is open when \mathcal{O}_F is given the subspace topology and the local field \overline{F} is endowed with its usual complete discrete valuation topology.

Remark 3.8. In order to show that $K((t))$ or $K\{\{t\}\}$ is Hausdorff, it suffices to show that given a nonzero element x , there is an admissible seminorm $\|\cdot\|$ for which $\|x\| \neq 0$. This is obvious.

Multiplication $\mu : F \times F \rightarrow F$ fails to be continuous when the product topology is considered on the left hand side. However, μ is separately continuous as explained above, and because the topology is a vector space topology, scalar multiplication $K \times F \rightarrow F$, which is compatible with μ , is continuous.

Another well known fact about higher topologies is that no basis of open neighbourhoods of zero is countable. In other words, these topologies do not satisfy the first countability axiom. This implies that the set of seminorms

defining the higher topology is uncountable. From the point of view of functional analysis, this shows that two-dimensional local fields are not Fréchet spaces, and in particular not metrizable.

Definition 3.9. We will call seminorms of the form (4) in the equal characteristic case and (7) in the mixed characteristic case *admissible*.

In both cases, admissible seminorms are attached to a sequence $(n_i)_{i \in \mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$, subject to different conditions, but satisfying the formula

$$\left\| \sum_i x_i t^i \right\| = \sup_i |x_i| q^{n_i};$$

the reasons why this formula is valid differ in each case.

Remark 3.10. Power series expressions of the form $x = \sum_i x_i t^i$ define convergent series with respect to the higher topology, in the sense that the net of partial sums $\left(\sum_{i \leq n} x_i t^i \right)_{n \in \mathbb{Z}}$ converges to x . If we let $x_n = \sum_{i \leq n} x_i t^i$ and $\|\cdot\|$ be any admissible seminorm, then

$$\|x - x_n\| = \left\| \sum_{i > n} x_i t^i \right\|$$

may be shown to be arbitrarily small if n is large enough.

Another well-known fact is that rings of integers $K[[t]]$ and $\mathcal{O}(\{t\})$ are closed but not open. In the first case, consider the set of open (and closed) lattices

$$\Lambda_n = \sum_{i \leq 0} \mathfrak{p}^{n t^i} + K[[t]], \quad n \geq 0$$

to find that $K[[t]] = \bigcap_{n \geq 0} \Lambda_n$ is closed. In the second case, consider the open (and closed) lattices:

$$\Lambda_n = \sum_{i < n} K t^i + \mathcal{O} t^n + \sum_{i > n} K t^i, \quad n \in \mathbb{Z}$$

and obtain that $\mathcal{O}(\{t\}) = \bigcap_{n \in \mathbb{Z}} \Lambda_n$ is closed. In order to see that these rings are not open, it is enough to say that they do not contain any open lattice.

4 Bounded sets and bornology

Let us describe the nature of bounded subsets of $K((t))$ and $K(\{t\})$. We will supply a description of a basis for the bornology of these fields, in the sense of [5], i.e.: we will give a family of bounded sets such that every bounded set is contained in a set of the family.

Example 4.1. Let $\|\cdot\|$ be an admissible seminorm, attached to the integers $\{n_i\}_{i \in \mathbb{Z}}$. The values of $\|\cdot\|$ on \mathcal{O} only depend on n_0 . If $n_0 = -\infty$ then the restriction of $\|\cdot\|$ to \mathcal{O} is identically zero. Otherwise, for any $x \in \mathcal{O}$ we have $\|x\| \leq q^{n_0}$ and therefore \mathcal{O} is bounded.

Similarly, if $n_0 > -\infty$, we may find elements $x \in K$ making the value $|x|q^{n_0}$ arbitrarily large. Hence, K is unbounded.

Proposition 4.2. *Let $\{k_i\}_{i \in \mathbb{Z}} \subset \mathbb{Z} \cup \{\infty\}$ such that there is an index $i_0 \in \mathbb{Z}$ such that $k_i = \infty$ for every $i < i_0$. The bornology of $K((t))$ admits the collection of sets given by*

$$B = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{k_i} t^i \quad (8)$$

as a basis, as $\{k_i\}_{i \in \mathbb{Z}}$ describes the collections specified above.

Proof. First, the subset B given by (8) is bounded: suppose that $\|\cdot\|$ is an admissible seminorm on $K((t))$ given by $\{n_i\}_{i \in \mathbb{Z}}$ and that k is the index for which $n_i = -\infty$ for every $i > k$.

If $k < i_0$, then the restriction of $\|\cdot\|$ to B is identically zero. Otherwise, for $x = \sum_{i \geq i_0} x_i t^i \in B$,

$$\|x\| = \max_{i_0 \leq i \leq k} |x_i| q^{n_i} \leq \max_{i_0 \leq i \leq k} q^{n_i - k_i},$$

and the bound is uniform for $x \in B$ once $\|\cdot\|$ has been fixed.

Next, we study general bounded sets. From Example 4.1 we deduce that if a subset of $K((t))$ contains elements for which one coefficient can be arbitrarily large, then the subset is unbounded in F . We may describe a basis of bounded sets of K by taking the disks \mathfrak{p}^n for $n \in \mathbb{Z}$.

In order to show our claim, it is enough to show that bounded sets of $K((t))$ cannot have elements with nonzero coefficients of arbitrary small degree. In order to see this, consider a set of the form

$$C = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{k_i} t^i, \quad k_i \in \mathbb{Z} \cup \{\infty\}$$

and suppose that for every i_0 there exists an $i < i_0$ such that $k_i < \infty$.

Let

$$n_i = \begin{cases} -\infty, & k_i = \infty \text{ or } i > 0, \\ -i + k_i, & \text{otherwise;} \end{cases}$$

and consider the admissible seminorm $\|\cdot\|$ on $K((t))$ attached to this collection of integers. Denote by $(i_j)_{j \geq 0}$ the partial sequence of negative indices such that $k_{i_j} < \infty$.

Let $x_j = \pi_K^{k_{i_j}} t^{i_j} \in C$ for every $j \geq 0$. We have that

$$\|x_j\| = q^{n_{i_j} - k_{i_j}} = q^{-i_j} \quad \text{for every } j \geq 0,$$

which shows that C is unbounded. \square

Corollary 4.3. *If $\|\cdot\| : K((t)) \rightarrow \mathbb{R}$ is a seminorm which is bounded on bounded sets, then there is an index $i_0 \in \mathbb{Z}$ such that $\|t^i\| = 0$ for all $i \geq i_0$.*

Proof. Suppose that for every $i_0 \in \mathbb{Z}$ there is an $i \geq i_0$ such that $\|t^i\| \neq 0$. If i is such that $\|t^i\| > 0$, take $k_i \in \mathbb{Z}$ such that

$$q^{-k_i} \|t^i\| \geq q^i.$$

If i is such that $\|t^i\| = 0$, take $k_i = 0$. By Proposition 4.2, the set

$$B = \sum_{i \geq 0} \mathfrak{p}^{k_i} t^i$$

is bounded. Let $x_j = \pi_K^{k_j} t^j$ for every $j \geq 0$. We have that $\|x_j\| = q^{-k_j} \|t^j\|$. Our hypothesis implies that the sequence of real numbers $(\|x_j\|)_{j \geq 0}$ is unbounded, and therefore $\|\cdot\|$ is not bounded on B . \square

We will now show that $K((t))$ is bornological. In fact, we will show slightly more: we will find a rather general description of all seminorms on this space.

Theorem 4.4. *The field $K((t))$ is bornological.*

Proof. Let $\|\cdot\| : K((t)) \rightarrow \mathbb{R}$ be a seminorm which is bounded on bounded sets. By writing an element $x = \sum_{i_0 \leq i < 0} x_i t^i + y$ with $y \in K[[t]]$ and using that

$$\|x\| \leq \max(|x_{i_0}| \cdot \|t^{i_0}\|, \dots, |x_{-1}| \cdot \|t^{-1}\|, \|y\|),$$

it is clear that we may restrict ourselves to the case of a seminorm $\|\cdot\| : K[[t]] \rightarrow \mathbb{R}$.

The locally convex K -vector space $K[[t]]$ is isomorphic to $K^{\mathbb{N}}$. Hence, the seminorm $\|\cdot\|$ is continuous if and only if for every $i \geq 0$ the restriction

$$\|\cdot\|_{Kt^i} : Kt^i \rightarrow \mathbb{R}$$

is a continuous seminorm.

But there are only two seminorms on K regarded as a K -vector space: the absolute value and the trivial seminorm, which is identically zero. Both seminorms are continuous on K . \square

Proposition 4.5. *Consider a sequence $\{k_i\}_{i \in \mathbb{Z}} \subset \mathbb{Z} \cup \{\infty\}$ which is bounded below. The bornology of $K\{\{t\}\}$ admits the sets of the form*

$$B = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{k_i} t^i \tag{9}$$

as a basis.

Proof. First, let us show that B is bounded. Assume all the k_i in (9) are bounded below by some integer d . Let $\|\cdot\|$ be an admissible seminorm on $K\{\{t\}\}$ defined by a collection $\{n_i\}_{i \in \mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$. In particular, there is an integer c such that $n_i \leq c$ for every $i \in \mathbb{Z}$.

Then, if $\sum x_i t^i \in B$, we have that

$$\left\| \sum x_i t^i \right\| = \sup_i |x_i| q^{n_i} \leq q^{c-d},$$

and the bound is uniform on B once $\|\cdot\|$ has been fixed.

Next, we study general bounded sets. Again, from Example 4.1 we may deduce that a subset of $K\{\{t\}\}$ which contains elements with arbitrarily large coefficients cannot be bounded. Therefore, to prove our claim it is enough to show that a subset

$$C = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{k_i} t^i$$

with $k_i \in \mathbb{Z}$ not bounded below is not bounded in $K\{\{t\}\}$. We see this in two steps. Since the k_i are not bounded below, there is either a subsequence of

them which tends to infinity as i tends to $-\infty$ or a subsequence which tends to infinity as i tends to $+\infty$.

In the first case, suppose that $(k_{i_j})_{j \in \mathbb{Z}}$ is such a subsequence, that is: $k_{i_j} \rightarrow -\infty$ as $j \rightarrow -\infty$. Let

$$n_i = \begin{cases} 0, & \text{if } i \leq 0, \\ -\infty, & \text{if } i > 0. \end{cases}$$

Denote by $\|\cdot\|$ the admissible seminorm attached to $\{n_i\}_{i \in \mathbb{Z}}$, and let $\alpha_j = \pi_K^{k_{i_j}} t^{i_j}$ for every $j \leq 0$. The sequence of positive real numbers

$$\|\alpha_j\| = q^{-k_{i_j}}, \quad j \leq 0$$

is unbounded, and hence C is unbounded in such case.

In the second case, suppose we have a subsequence $k_{i_j} \rightarrow -\infty$ when $j \rightarrow \infty$. For ease of notation, we suppose that $i_0 = 0$. Let

$$n_{i_j} = \begin{cases} -\infty, & \text{if } j < 0, \\ \frac{k_{i_j}-1}{2} & \text{if } j \geq 0, k_{i_j} \text{ odd}, \\ \frac{k_{i_j}}{2}, & \text{if } j \geq 0, k_{i_j} \text{ even}. \end{cases}$$

Furthermore, for any index l such that $i_j \leq l < i_{j+1}$, let $n_l = n_{i_j}$. With such choices, the following three facts hold:

- (i) The sequence $(n_{i_j} - k_{i_j})_{j \geq 0}$ tends to infinity.
- (ii) The sequence $(n_i)_{i \in \mathbb{Z}}$ is bounded above.
- (iii) For any $l \in \mathbb{Z}$, there is an index i_0 such that $n_i \leq l$ for all $i \geq i_0$.

By (ii) and (iii), we may consider the admissible seminorm $\|\cdot\|$ associated to $\{n_i\}_{i \in \mathbb{Z}}$. Let $\alpha_j = \pi_K^{k_{i_j}} t^{i_j}$ for $j \geq 0$. By (i),

$$\|\alpha_j\| = q^{n_{i_j} - k_{i_j}}, \quad j \geq 0$$

is an unbounded sequence of positive real numbers and hence C is not bounded. \square

Corollary 4.6. *If $\|\cdot\| : K\{\{t\}\} \rightarrow \mathbb{R}$ is a seminorm which is bounded on bounded sets, then there is a real number $C > 0$ such that $\|t^i\| < C$ for every $i \in \mathbb{Z}$.*

Proof. Suppose that $\|\cdot\|$ is a seminorm such that the sequence of real numbers $\{\|t^i\|\}_{i \in \mathbb{Z}}$ is not bounded. Consider the bounded set

$$\mathcal{O}\{\{t\}\} = \sum_{i \in \mathbb{Z}} \mathcal{O}t^i,$$

and the sequence $\{t^i\}_{i \in \mathbb{Z}} \subset \mathcal{O}\{\{t\}\}$. The seminorm $\|\cdot\|$ is not bounded on \mathcal{O}_F . \square

Corollary 4.7. *The space $K\{\{t\}\}$ is not bornological.*

Proof. It is enough to supply a seminorm which is bounded on bounded sets but not continuous.

Consider the norm on $K\{\{t\}\}$ given by

$$\left\| \sum_{i \in \mathbb{Z}} x_i t^i \right\| = \sup_{i \in \mathbb{Z}} |x_i|, \quad (10)$$

which is the absolute value on $K\{\{t\}\}$ related to the valuation v_F . If B is a basic bounded set as in (9), and $x \in B$, then

$$\sup_{x \in B} \|x\| = \sup_{i \in \mathbb{Z}} q^{-k_i},$$

and hence $\|\cdot\|$ is bounded on bounded sets. However, the norm $\|\cdot\|$ is not continuous on $K\{\{t\}\}$ because

$$\mathcal{O}\{\{t\}\} = \{x \in K\{\{t\}\}; \|x\| \leq 1\}$$

is not open in $K\{\{t\}\}$. \square

Proposition 4.8. *Let $F = K((t))$ or $K\{\{t\}\}$. The multiplication map $\mu : F \times F \rightarrow F$ is bounded (with respect to the product bornology in the domain).*

Proof. Let $B_1 = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{m_i} t^i$ and $B_2 = \sum_{j \in \mathbb{Z}} \mathfrak{p}^{n_j} t^j$ be two bounded \mathcal{O} -submodules of F . The product bornology on $F \times F$ is generated by sets of the form $B_1 \times B_2$. We have that $\mu(B_1, B_2) = \sum_{k \in \mathbb{Z}} V_k t^k$ with $V_k = \sum_{i+j=k} \mathfrak{p}^{m_i} \mathfrak{p}^{n_j} = \sum_{k=i+j} \mathfrak{p}^{m_i+n_j}$. We distinguish cases.

If $F = K((t))$, $m_i = \infty$ and $n_j = \infty$ if i and j are small enough. In this case, the sum defining V_k is actually finite and there is $l_k \in \mathbb{Z} \cup \{\infty\}$ such that $V_k \subset \mathfrak{p}^{l_k}$. Moreover, we actually have $V_k = \{0\}$ if k is small enough and therefore $\mu(B_1, B_2) \subset F$ is bounded.

If $F = K\{\{t\}\}$, then there are integers c and d such that $m_i \geq c$ for all $i \in \mathbb{Z}$ and $n_j \geq d$ for all $j \in \mathbb{Z}$. This implies that $V_k \subset \mathfrak{p}^{c+d}$ for every k and that it is bounded. \square

5 Complete, c -compact and compactoid \mathcal{O} -submodules

In this section we will study relevant \mathcal{O} -submodules of $K((t))$ and $K\{\{t\}\}$, including rings of integers and rank-2 rings of integers. The submodules we will study are complete for nets, and we will establish which ones of them are c -compact and which ones are compactoid.

We start dealing with completeness of rings of integers.

Proposition 5.1. *The rings of integers $K[[t]]$ and $\mathcal{O}\{\{t\}\}$ are complete \mathcal{O} -submodules of $K((t))$ and $K\{\{t\}\}$, respectively.*

Proof. Let I be a directed set and $(x_i)_{i \in I}$ a Cauchy net in the ring of integers. We distinguish cases below.

If $\mathcal{O}_F = K[[t]]$, we write $x_i = \sum_{k \geq 0} x_{k,i} t^k$ with $x_{k,i} \in K$. Since $(x_i)_{i \in I}$ is a Cauchy net in \mathcal{O}_F , we have that $(x_{k,i})_{i \in I}$ is a Cauchy net in K and hence

converges to an element $x_k \in K$ for every $k \geq 0$. The element $x = \sum_{k \geq 0} x_k t^k$ is the limit of the Cauchy net.

If $\mathcal{O}_F = \mathcal{O}\{\{t\}\}$, the procedure is very similar. We write $x_i = \sum_{k \in \mathbb{Z}} x_{i,k} t^k$ with $x_{i,k} \in \mathcal{O}$. Since \mathcal{O} is complete and $(x_{i,k})_{i \in I}$ is a Cauchy net, it converges to an element $x_k \in \mathcal{O}$ for every $k \in \mathbb{Z}$. It is elementary to check that as $k \rightarrow -\infty$, we have $x_k \rightarrow 0$ and therefore $x = \sum_{k \in \mathbb{Z}} x_k t^k$ is a well-defined element in $\mathcal{O}\{\{t\}\}$ which is the limit of the Cauchy net. \square

Corollary 5.2. *The rank-2 rings of integers of $K((t))$ and $K\{\{t\}\}$ are complete.*

Proof. It follows from the previous proposition due to the fact that they are closed subsets of complete \mathcal{O} -submodules. \square

Next we will study rings of integers from the point of view of c-compactness and compactoidness.

Proposition 5.3. *$K[[t]]$ is c-compact.*

Proof. As a locally convex K -vector space, $K[[t]]$ is isomorphic to $K^{\mathbb{N}}$ (Proposition 3.4). The field K is c-compact (Example 1.6). Finally, a product of c-compact spaces is c-compact (Proposition 1.8). \square

Corollary 5.4. *The rank-2 ring of integers of $K((t))$, $\mathcal{O} + tK[[t]]$, is c-compact.*

Proof. After the previous proposition, the result follows from the fact that $\mathcal{O} + tK[[t]] \subset K[[t]]$ is closed, as c-compactness is hereditary for closed subsets [13, Lemma 12.1.iii]. \square

Corollary 5.5. *The rings $K[[t]]$ and $\mathcal{O} + tK[[t]]$ are not compactoid.*

Proof. This follows from the fact that they are both c-compact, unbounded, complete and Proposition 1.10. \square

The situation for $K\{\{t\}\}$ is much different. From Proposition 1.10, since $\mathcal{O}\{\{t\}\}$ is complete and bounded, it is c-compact if and only if it is compactoid. The case is that is it neither.

Proposition 5.6. *$\mathcal{O}\{\{t\}\}$ is not compactoid.*

Proof. It is enough to exhibit an open lattice Λ in $K\{\{t\}\}$ such that no finite number of elements $x_1, \dots, x_m \in K\{\{t\}\}$ satisfy

$$\mathcal{O}\{\{t\}\} \subset \Lambda + \mathcal{O}x_1 + \dots + \mathcal{O}x_m.$$

Choose, for example,

$$n_i = \begin{cases} -\infty, & \text{for } i \in \mathbb{Z}, i \geq 0, \\ 1, & \text{for } i < 0. \end{cases}$$

The open lattice $\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i$ is a suitable such lattice. \square

Corollary 5.7. *The rank-2 ring of integers of $K\{\{t\}\}$ is not compactoid.*

Proof. By writing the rank-2 ring of integers as $\sum_{i < 0} \mathfrak{p}t^i + \sum_{i \geq 0} \mathcal{O}t^i$, the argument in the proof of the previous proposition also works in this case, choosing Λ in a similar fashion. \square

Finally, we conclude this section with the description of all compactoid submodules, which are to be found amongst bounded submodules.

Proposition 5.8. *The sets of the form (8), which constitute a basis for the bornology of $K((t))$, are the only compactoid \mathcal{O} -submodules of $K((t))$.*

Proof. The sets

$$B = \sum_{i \geq i_0} \mathfrak{p}^{k_i} t^i, \quad k_i \in \mathbb{Z} \cup \{\infty\}$$

are the only bounded \mathcal{O} -submodules of $K((t))$. Suppose that $\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i$ with $n_i \in \mathbb{Z} \cup \{-\infty\}$ and such that for every $i > i_1$ we have $n_i = -\infty$.

If $i_1 < i_0$ then $B \subset \Lambda$ and there is nothing to show. Otherwise, let $l_i = \min(n_i, k_i)$ for $i_0 \leq i \leq i_1$. Then

$$B \subseteq \Lambda + \sum_{i=i_0}^{i_1} \mathcal{O} \cdot \pi^{l_i} t^i,$$

which shows that it is compactoid. \square

Corollary 5.9. *The compactoid \mathcal{O} -submodules of $K((t))$ are c-compact.*

Proof. In view of Proposition 1.10, it is enough to show that a submodule B as in the proof of the previous proposition is complete for nets. But the argument for showing completeness of such \mathcal{O} -submodules is the same as in the proof of Proposition 5.1 and we shall omit it. \square

In the case of $K\{\{t\}\}$ there is a difference between bounded and compactoid \mathcal{O} -submodules; we have already shown how $\mathcal{O}\{\{t\}\}$ is a bounded \mathcal{O} -submodule which is neither c-compact nor compactoid.

Proposition 5.10. *The only compactoid submodules of $K\{\{t\}\}$ are the ones of the form*

$$B = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{k_i} t^i, \tag{11}$$

with $k_i \in \mathbb{Z}$ bounded below and such that $k_i \rightarrow \infty$ as $i \rightarrow -\infty$.

Proof. In view of Proposition 4.5, all bounded submodules of $K\{\{t\}\}$ are of the form (11) with the $k_i \in \mathbb{Z} \cup \{\infty\}$ bounded below. We only have to show that B is compactoid if and only if $k_i \rightarrow \infty$ as $i \rightarrow -\infty$.

Assume that $k_i \rightarrow \infty$ as $i \rightarrow -\infty$. Let $\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i$ be an open lattice and assume that B is not contained in Λ , as otherwise there is nothing to prove. When $i \rightarrow \infty$, the k_i are bounded below and $n_i \rightarrow -\infty$. Similarly, when $i \rightarrow -\infty$, $k_i \rightarrow \infty$ and the n_i are bounded below. Hence, the following two statements are true:

- (i) There is an index i_0 such that for every $i < i_0$, $k_i \geq n_i$.
- (ii) There is an index i_1 such that for every $i > i_1$, $k_i \geq n_i$.

We have $i_0 \leq i_1$, as otherwise B is contained in Λ . Let $l_i = \min(k_i, n_i)$ for $i_0 \leq i \leq i_1$. Then we have

$$B \subseteq \Lambda + \sum_{i=i_0}^{i_1} \mathcal{O} \cdot \pi^{l_i} t^i,$$

which shows that B is compactoid.

Reciprocally, it is possible to show that if the k_i do not tend to infinity as $i \rightarrow -\infty$ then there are lattices Λ such that no finite number of elements $x_1, \dots, x_m \in K\{\{t\}\}$ yields an inclusion

$$B \subseteq \Lambda + \mathcal{O}x_1 + \dots + \mathcal{O}x_m.$$

□

Corollary 5.11. *The compactoid submodules of $K\{\{t\}\}$ are c -compact.*

Proof. Again, in view of Proposition 1.10, it is enough to show that the \mathcal{O} -submodule B as in (11) is complete. The argument is the same as in the proof of Proposition 5.1 and we omit it. □

Corollary 5.12. *Multiplication $\mu : K\{\{t\}\} \times K\{\{t\}\} \rightarrow K\{\{t\}\}$ is also bounded when $K\{\{t\}\}$ is endowed with the bornology generated by compactoid \mathcal{O} -submodules in the codomain, and the product of two copies of such bornology in the domain.*

Proof. Let $B_1 = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{m_i} t^i$ and $B_2 = \sum_{j \in \mathbb{Z}} \mathfrak{p}^{n_j} t^j$ be two compactoid \mathcal{O} -submodules of $K\{\{t\}\}$; let V_k as in the proof of Proposition 4.8. Just like in the aforementioned proof, V_k is contained in a fractional ideal of K and is therefore bounded. Moreover, it is possible to choose $l_k \in \mathbb{Z} \cup \{\infty\}$ such that $l_k \rightarrow \infty$ as $k \rightarrow -\infty$ and $V_k \subset \mathfrak{p}^{l_k}$; this proves that $\mu(B_1, B_2)$ is contained in a compactoid \mathcal{O} -submodule of $K\{\{t\}\}$. □

6 Duality

Let us describe some duality issues of two-dimensional local fields when regarded as locally convex vector spaces over a local field.

Much is known about the self-duality of the additive group of a two-dimensional local field. From [3, §3], if F is a two-dimensional local field, once a nontrivial continuous character

$$\psi : F \rightarrow S^1 \subset \mathbb{C}^\times$$

has been fixed, the group of continuous characters of the additive group of F consists entirely of characters of the form $\alpha \rightarrow \psi(a\alpha)$, where a runs through all elements of F . This result is entirely analogous to the one-dimensional theory [14, Lemma 2.2.1].

In the case of $K((t))$ and $K\{\{t\}\}$, self-duality of the additive group follows in an explicit way from two self-dualities: that of the two-dimensional local field as a locally convex K -vector space, and that of the additive group of K as a locally compact abelian group. Since the second is sufficiently well-known [14, §2.2], let us focus on the first one.

We can exhibit nontrivial continuous linear forms on a two-dimensional local field effortlessly. Let $F = K((t))$ or $K\{\{t\}\}$.

Example 6.1. The map

$$\pi_i : F \rightarrow K, \quad \sum x_j t^j \mapsto x_i \quad (12)$$

is a continuous nonzero linear form for all $i \in \mathbb{Z}$.

Consider now the following map:

$$\gamma : F \rightarrow F', \quad x \mapsto \pi_x,$$

with

$$\pi_x : F \rightarrow K, \quad y \mapsto \pi_0(xy).$$

More explicitly, if $x = \sum x_i t^i$ and $y = \sum y_i t^i$, then

$$\pi_x(y) = \sum x_i y_{-i}.$$

The map γ is well-defined because π_x , being the composition of multiplication by a fixed element $F \rightarrow F$ and the projection $\pi_0 : F \rightarrow K$, is a continuous linear form. Besides that, γ is K -linear and injective.

Remark 6.2. Regarding topologies on dual spaces, we have that $K((t))'_c = K((t))'_b$ (Proposition 5.8). However, the topology of $K\{\{t\}\}'_c$ is strictly weaker than the one of $K\{\{t\}\}'_b$.

Theorem 6.3. *The map $\gamma : F \rightarrow F'_c$ is an isomorphism of locally convex K -vector spaces.*

Before we prove this result, we need an auxiliary result.

Lemma 6.4. *Let $w \in F'$ and define, for every $i \in \mathbb{Z}$, $a_i = w(t^{-i})$. Then the formal sum $\sum a_i t^i$ defines an element of F .*

Proof. We distinguish cases. If $F = K[[t]]$, it is necessary to show that $a_i = 0$ for all small enough indices i . In other words, that there is an index $i_0 \in \mathbb{Z}$ such that for every $i \geq i_0$ we have $w(t^i) = 0$. Without loss of generality, we may restrict ourselves to a linear form $w : K[[t]] \rightarrow K$. As the isomorphism $K[[t]] \simeq K^\mathbb{N}$ is both algebraic and topological, we see that a linear form $K^\mathbb{N}$ is nothing but a single linear relation involving finitely many variables. Hence, we get our result in this case.

In the case in which $F = K\{\{t\}\}$, we need to show that the values $|a_i|$ for $i \in \mathbb{Z}$ are bounded and that $|a_i| \rightarrow 0$ as $i \rightarrow -\infty$. On one hand, the subset $\mathcal{O}\{\{t\}\} \subset F$ is bounded and $t^i \in \mathcal{O}\{\{t\}\}$ for every $i \in \mathbb{Z}$. As w is continuous, the set $w(\mathcal{O}\{\{t\}\}) \subset K$ is bounded and therefore the values $w(t^i)$ are bounded. On the other hand, the net $(t^i)_{i \in \mathbb{Z}}$ tends to zero in $K\{\{t\}\}$ as $i \rightarrow \infty$. As w is continuous, $a_i = w(t^{-i}) \rightarrow 0$ as $i \rightarrow -\infty$. \square

Proof of Theorem 6.3. As explained above, the map γ is well-defined, K -linear and injective.

Let $w \in F'$. Define $x = \sum_i a_i t^i \in F$ with a_i as in Lemma 6.4. Then, for $y = \sum y_i t^i \in F$, we have

$$w\left(\sum y_i t^i\right) = \sum y_i w(t^i) = \sum y_i a_{-i} = \pi_0(xy)$$

(the first equality follows from Remark 3.10). Therefore, $w = \pi_x$ and the map δ is surjective.

In order to show bicontinuity, let us first work out what continuity means in this setting. For any $\varepsilon > 0$ and B a set in the bornology generated by compactoid submodules, we must show that there are $\delta > 0$ and an admissible seminorm $\|\cdot\| : F \rightarrow K$ such that $\|x\| \leq \delta$ implies $|\pi_x|_B \leq \varepsilon$.

Without loss of generality, we may replace ε and δ by integer powers of q , and the generic bounded set B by a compactoid submodule of F , which is of the form (8) in the equal characteristic case or (11) in the mixed characteristic case. For convenience, let us write

$$B = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{k_i} t^i, \in \mathbb{Z} \cup \{\infty\}$$

by allowing, in the equal characteristic case, $k_i = \infty$ for every small enough i .

Now, let $n \in \mathbb{Z}$. We take $n_i = -k_{-i}$ for every $i \in \mathbb{Z}$. Because of the conditions defining the bounded sets which have been taken into consideration, the sequence $\{n_i\}_{i \in \mathbb{Z}}$ defines an admissible seminorm $\|\cdot\|$ in both cases. Now, for $x = \sum x_i t^i$, we have that $\|x\| \leq q^n$ if and only if for every index $i \in \mathbb{Z}$ we have

$$n_i - n \leq v_K(x_i). \quad (13)$$

Similarly, $|\pi_x|_B \leq q^n$ if and only if for every index $i \in \mathbb{Z}$ we have

$$-k_{-i} - n \leq v_K(x_i). \quad (14)$$

By direct comparison and substitution between (13) and (14), we have that with our choice of admissible seminorm $\|\cdot\|$,

$$\|x\| \leq q^n \text{ if and only if } |\pi_x|_B \leq q^n,$$

which shows bicontinuity. \square

Remark 6.5. In the mixed characteristic case we may ask ourselves if it is possible to exhibit any self-duality result involving F'_b , that is, topologizing the dual space according to uniform convergence over all bounded sets.

It can be seen from the proof of Theorem 6.3 that this is not the case. Any bornology \mathcal{B} stronger than the one generated by compactoid submodules will stop the map $\gamma : F \rightarrow F'_\mathcal{B}$ from being continuous.

We remark that if there were no other bounded sets in $K\{\{t\}\}$ besides the ones generated by compactoid submodules, it would be possible to show that such a locally convex vector space is bornological.

From the failure of $K\{\{t\}\}$ at being bornological we may deduce that Theorem 6.3 is the best result we can achieve.

The choice of a family of bounded subsets \mathcal{B} of F does not affect $F'_\mathcal{B}$ as a set, but it does affect the bidual space. As such, in the category of locally convex vector spaces over K , it is an interesting issue to classify which spaces are isomorphic, algebraically and/or topologically, to certain bidual spaces through the duality maps

$$\delta : V \rightarrow (V'_\mathcal{B})', \quad v \mapsto \delta_v(l) = l(v). \quad (15)$$

The best possible case is when δ induces a topological isomorphism between V and $(V'_b)'_b$; in this case we say that V is reflexive.

The following corollaries follow from Theorem 6.3.

Corollary 6.6. *The field $K((t))$ is a reflexive locally convex K -space.* \square

Corollary 6.7. *The field $K\{\{t\}\}$ is not reflexive.*

Proof. We deduce that δ induces an isomorphism $F \simeq (F'_c)'_c$. However, the inclusion $(F'_c)' \subset (F'_b)'$ is strict. \square

Proposition 6.8. *The field $K((t))$ is barrelled.*

Proof. It follows from reflexivity [13, §15]. \square

We also take the chance to say something about barrelledness of $K\{\{t\}\}$ here.

Proposition 6.9. *The field $K\{\{t\}\}$ is not barrelled.*

Proof. The ring of integers $\mathcal{O}\{\{t\}\}$ is a lattice which is closed but not open (cf. proof to Theorem 4.4). \square

In order to conclude this section let us describe polars and pseudopolars of the \mathcal{O} -submodules which we have studied in §5.

After Theorem 6.3, the topological isomorphism given by γ allows us to identify F with F'_c , and in particular lets us relate their \mathcal{O} -submodules. In particular, let $A \subset F$ be any \mathcal{O} -submodule.

Definition 6.10. Let $F = K((t))$ or $K\{\{t\}\}$. Let $A \subset F$ be an \mathcal{O} -submodule. We let

$$A^\gamma = \gamma^{-1}(A^p) \subset F$$

and refer to it, by abuse of language, as the pseudopolar of A .

Proposition 6.11. *Suppose that*

$$A = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{k_i} t^i, \quad k_i \in \mathbb{Z} \cup \{\pm\infty\}$$

is an \mathcal{O} -submodule of $F = K((t))$ or $K\{\{t\}\}$. Then, we have

$$A^\gamma = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{1-k_{-i}} t^i.$$

Proof. Let $B = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{1-k_{-i}} t^i$.

On one hand, suppose $x = \sum x_i t^i \in B$. We have, for every $y = \sum y_i t^i \in A$,

$$|\pi_x(y)| = \left| \sum x_{-i} y_i \right| \leq \sup |x_{-i}| |y_i| = \sup q^{-1+k_i-k_i} < 1$$

and, therefore, $B \subseteq A^\gamma$.

On the other hand, suppose that $x = \sum x_i t^i \in A^\gamma$. Then, by definition, we have

$$|\pi_x(y)| < 1, \quad \text{for any } y \in A.$$

In particular, let $y = \pi^{k_i} t^i$. Then the inequality

$$|\pi_x(\pi^{k_i} t^i)| = |x_{-i} \pi^{k_i}| < 1$$

implies that $v_K(x_{-i}) \geq 1 - k_i$. Therefore $x_{-i} \in \mathfrak{p}^{1-k_i}$. Since our conclusion holds for any $i \in \mathbb{Z}$, we have that $x \in A^\gamma$ and, therefore, $B \subset A^\gamma$. \square

Corollary 6.12. *For an \mathcal{O} -submodule A as in the previous Proposition, we have $A^{pp} = A$.* \square

Corollary 6.13. *We have $K[[t]]^\gamma = K[[t]]$. For the rank-2 ring of integers, we have $(\mathcal{O} + tK[[t]])^\gamma = \mathfrak{p} + tK[[t]]$.* \square

Corollary 6.14. *We have $(\mathcal{O}\{\{t\}\})^\gamma = \mathfrak{p}\{\{t\}\}$. For the rank-2 ring of integers, we have $(\sum_{i<0} \mathfrak{p}t^i + \sum_{i\geq 0} \mathcal{O}t^i)^\gamma = \sum_{i\leq 0} \mathcal{O}t^i + \sum_{i>0} \mathfrak{p}t^i$.* \square

By Proposition 1.13 and Theorem 6.3, pseudopolarity exchanges open lattices and compactoid submodules. Under the characterization given by Proposition 6.11, the relation is evident.

Although the description of A^γ for an \mathcal{O} -submodule A is intimately linked to the choice of the isomorphism γ which is not unique, the relation between open lattices and compactoid submodules given by polarity does not depend on the choice of an isomorphism $F \simeq F'_c$.

The same arguments exposed apply to compute that the polar of the \mathcal{O} -submodule $\sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i} t^i$, $k_i \in \mathbb{Z} \cup \{\pm\infty\}$ is $\sum_{i\in\mathbb{Z}} \mathfrak{p}^{-k-i} t^i$. As such, the polar of an open lattice is a compactoid lattice and viceversa.

Let us write down a table with pseudopolars and polars of relevant \mathcal{O} -submodules:

\mathbf{A}	\mathbf{A}^γ	polar of \mathbf{A}
$K[[t]]$	$K[[t]]$	$K[[t]]$
$\mathcal{O} + tK[[t]]$	$\mathfrak{p} + tK[[t]]$	$\mathcal{O} + tK[[t]]$
$\mathcal{O}\{\{t\}\}$	$\mathfrak{p}\{\{t\}\}$	$\mathcal{O}\{\{t\}\}$
$\sum_{i<0} \mathfrak{p}t^i + \sum_{i\geq 0} \mathcal{O}t^i$	$\sum_{i\leq 0} \mathcal{O}t^i + \sum_{i>0} \mathfrak{p}t^i$	$\sum_{i\leq 0} \mathcal{O}t^i + \sum_{i>0} \mathfrak{p}^{-1}t^i$
$\Lambda = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{n_i} t^i$ (open lattice)	$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{1-n-i} t^i$ (compactoid)	$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{-n-i} t^i$ (compactoid)
$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i} t^i$ (compactoid)	$\Lambda = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{1-k-i} t^i$ (open lattice)	$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{-k-i} t^i$ (open lattice)

In conclusion, taking the pseudopolar or polar is a self-map on the set of \mathcal{O} -submodules of $K[[t]]$ or $K\{\{t\}\}$ which reverses inclusions, gives compactoid submodules when applied to open lattices and viceversa, and whose square equals the identity map.

7 Nuclearity

The class of nuclear spaces plays a relevant role in the theory of locally convex vector spaces. As has been the case in previous matters of study, we get different behaviours in equal characteristic and mixed characteristic: in the first case we get our result from the structure of $K((t))$ as a direct limit of Fréchet spaces and in the second case we get a counterexample from the properties of $\mathcal{O}\{\{t\}\}$.

Proposition 7.1. *The field $K((t))$ is nuclear.*

Proof. On one hand, K is trivially nuclear because it is a finite-dimensional K -vector space. The field $K((t))$ is a strict inductive limit

$$K((t)) = \varinjlim_{i \in \mathbb{N}} t^{-i} K[[t]]$$

of locally convex vector spaces (cf. Remark 3.5), each of them being isomorphic to a countable direct product of copies K . Nuclearity is preserved under arbitrary direct products of locally convex spaces by Proposition 1.15.ii, and also under strict inductive limits by Proposition 1.15.iii. \square

Proposition 7.2. *The field $K\{\{t\}\}$ is not nuclear.*

Proof. According to Proposition 1.15.i, the class of bounded \mathcal{O} -submodules and the class of compactoid \mathcal{O} -submodules agree for nuclear spaces. The ring of integers $\mathcal{O}\{\{t\}\}$ is an example of an \mathcal{O} -submodule which is bounded but not compactoid. \square

After Propositions 1.15.i and 7.1, we obtain that Proposition 5.8 follows from nuclearity of $K((t))$. We have however chosen to give a direct explicit proof for that result.

8 General two-dimensional local fields

In the previous sections of this work we have developed a systematic study of $K((t))$ and $K\{\{t\}\}$ from the point of view of the theory of locally convex spaces over K . Let us explain how the previous results extend to a general characteristic zero two-dimensional local field $K \hookrightarrow F$. Due to the difference in their structures, we consider the equal characteristic and mixed characteristic cases separately.

8.1 Equal characteristic

Assume that $K \hookrightarrow F$ is a two-dimensional local field and that $\text{char } F = \text{char } \overline{F}$. In this case, as explained in §2, the choice of a field embedding $\overline{F} \hookrightarrow F$ determines an isomorphism $F \simeq \overline{F}((t))$.

Denote the algebraic closure of K in F by \tilde{K} . The extension $\tilde{K}|K$ is finite and $\tilde{K} \hookrightarrow F$ is the only coefficient field of F which factors the field inclusion $K \hookrightarrow F$, and this is the only coefficient field of F that we will take into account in our constructions.

Remark 8.1. It is a well-known fact that in this case the higher topology of F depends on the choice of a coefficient field. This is why we stress that in this work the only coefficient field we consider is $\tilde{K} \hookrightarrow F$ because the field embedding $K \hookrightarrow F$ is given a priori.

The \tilde{K} -vector space $F \simeq \tilde{K}((t))$ is a complete bornological nuclear locally convex space by direct application of the previous sections of this work. We verify that these properties do not change if we restrict scalars along the inclusion $K \hookrightarrow \tilde{K}$.

On one hand, all open lattices Λ are $\mathcal{O}_{\tilde{K}}$ -modules and hence also \mathcal{O} -modules by restriction of scalars. On the other hand, if $x \in F$, there is a positive power

of $\pi_{\tilde{K}}$ that maps x to Λ by multiplication. The uniformizers of K and \tilde{K} are related by the ramification degree: $\pi = \pi_{\tilde{K}}^{e(\tilde{K}|K)}$. Therefore, there is a positive power of π which maps x to Λ by multiplication. So we deduce local convexity over K .

The absolute value on \tilde{K} restricts to the absolute value of K and therefore Corollary 3.2 describes the admissible seminorms of F without any changes.

Because seminorms do not change after restricting scalars to K , Proposition 4.2 describes bounded \mathcal{O} -submodules of F . These are complete, and Proposition 5.8 applies to show that bounded \mathcal{O} -submodules of F agree with compactoid \mathcal{O} -submodules.

We have therefore that F is bornological. \mathcal{O}_F is c -compact, unbounded, complete and not compactoid. Similarly, the rank-2 ring of integers of F inherits these properties from \mathcal{O}_F .

Regarding duality, the fact that the map $\gamma : F \rightarrow F'_c$ is an isomorphism of locally convex spaces does not change when we restrict scalars to K and hence F is reflexive. Explicit nonzero linear forms $F \rightarrow K$ may be constructed by composing the maps $\pi_i : F \rightarrow \tilde{K}$ as in (12) with $\text{Tr}_{\tilde{K}|K}$. Finally, F is also nuclear.

Remark 8.2. It is a relevant question to decide whether the bornology of F might change along with the higher topology if we choose a different coefficient field. The answer is that the bornology also changes in such case because F is bornological.

If the bornology is given, the higher topology is the strongest locally convex topology on F which yields the given collection of bounded sets. If the bornology did not depend on choosing a coefficient field, neither would the topology. This is not the case.

8.2 Mixed characteristic

If $\text{char } F \neq \text{char } \overline{F}$, then the field inclusion $K \hookrightarrow F$ may be factored into

$$K \hookrightarrow K\{\{t\}\} \hookrightarrow F,$$

with the field extension $F|K\{\{t\}\}$ being finite. The inclusion $K \hookrightarrow K\{\{t\}\}$ is the situation we have been dealing with in all the previous sections of this work. Let $n = [F : K\{\{t\}\}]$.

As K -vector spaces, we have

$$F \simeq K\{\{t\}\}^n.$$

The higher topology on F may be defined as the product topology on n copies of the higher topology on $K\{\{t\}\}$ [9, 1.3.2]. Furthermore, it does not depend on any choices of subfields $K\{\{t\}\} \subset F$ [7, §1].

The product topology on a cartesian product of locally convex K -vector spaces is again locally convex. So we may describe the family of open lattices or, equivalently, continuous seminorms, from the corresponding lattices or seminorms for $K\{\{t\}\}$ and Definition 1.1.

The situation for the ring of integers \mathcal{O}_F is similar. The inclusion $\mathcal{O}\{\{t\}\} \hookrightarrow \mathcal{O}_F$ turns \mathcal{O}_F into a rank- n free $\mathcal{O}\{\{t\}\}$ -module. Therefore the subspace topology on $\mathcal{O}_F \subset F$ coincides with the product topology on $\mathcal{O}_F \simeq \mathcal{O}\{\{t\}\}^n$. From here,

it is possible to show that \mathcal{O}_F is a bounded and complete \mathcal{O} -submodule of F which is neither c -compact nor compactoid. The norm attached to the valuation v_F provides an example of a seminorm which is bounded on bounded sets but not continuous, as \mathcal{O}_F is not open in F . Hence F is not bornological.

From the self-duality of $K\{\{t\}\}$, we obtain a chain of isomorphisms

$$F'_c \simeq (K\{\{t\}\}^n)'_c \simeq (K\{\{t\}\}'_c)^n \simeq K\{\{t\}\}^n \simeq F,$$

which shows that F is also self-dual. Explicit nonzero linear forms may be constructed in this case composing the trace map $\text{Tr}_{F|K\{\{t\}\}}$ with the maps $\pi_i : K\{\{t\}\} \rightarrow K$ as in (12). Finally, as \mathcal{O}_F is a bounded \mathcal{O} -submodule which is not compactoid, we deduce that F is not nuclear.

9 A note on the archimedean case

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . We will denote by $|\cdot|$ either the usual absolute value on \mathbb{R} , or the module on \mathbb{C} .

In this section we will consider the study of archimedean two-dimensional local fields. An archimedean two-dimensional local field is a complete discrete valuation field F whose residue field is an archimedean (one-dimensional) local field. Hence, we have a non-canonical isomorphism $F \simeq \mathbb{K}((t))$ for one of our two choices of \mathbb{K} . Once an inclusion of fields $\mathbb{K} \subset F$ has been fixed and t has been chosen, a unique such isomorphism is determined.

The theory of locally convex vector spaces over \mathbb{K} was developed much earlier than the analogous non-archimedean theory and is well explained in, for example, [8]. Let V be a \mathbb{K} -vector space and $C \subseteq V$. The subset C is said to be convex if for any $v, w \in C$, the segment

$$\{\lambda v + \mu w; \lambda, \mu \in \mathbb{R}_{\geq 0}, \lambda + \mu = 1\}$$

is contained in C . The subset C is said to be absolutely convex if, moreover, we have $\lambda C \subseteq C$ for every $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$.

We may associate a seminorm p_C to any convex subset $C \subseteq V$ by the rule

$$p_C : V \rightarrow \mathbb{R}, \quad x \mapsto \inf_{\rho > 0, x \in \rho C} \rho.$$

This seminorm satisfies the usual triangle inequality, but not the ultrametric inequality.

Definition 9.1. The \mathbb{K} -vector space V is said to be locally convex if it is a topological vector space such that its topology admits a basis of neighbourhoods of zero given by convex sets.

It may be shown that if V is locally convex its filter of neighbourhoods of zero also admits a basis formed by absolutely convex subsets [8, §18.1].

The higher topology on $\mathbb{K}((t))$ is defined following the procedure outlined in §3.1. In this case, we consider the disks of \mathbb{K} centered at zero and of rational radius; this defines a countable basis of convex neighbourhoods of zero for the euclidean topology on \mathbb{K} . Denote

$$D_\rho = \{a \in \mathbb{K}; |a| < \rho\}, \quad \rho \in \mathbb{Q}_{>0} \cup \{\infty\}.$$

Given a sequence $(\rho_i)_{i \in \mathbb{Z}} \subset \mathbb{Q}_{>0} \cup \{\infty\}$ such that there is an index i_0 satisfying that $\rho_i = \infty$ for all $i \geq i_0$, consider the set

$$\mathcal{U} = \sum_{i \in \mathbb{Z}} D_{\rho_i} t^i \subset F. \quad (16)$$

The sets of the form (16) form a basis of neighbourhoods of zero for the higher topology on F .

Proposition 9.2. *The higher topology on F is locally convex, in the sense of Definition 9.1.*

Proof. As the discs D_{ρ_i} are convex, given two elements $x, y \in \mathcal{U}$, it is easy to check that the segment

$$\{\lambda x + \mu y; \lambda, \mu \in \mathbb{R}_{\geq 0}; \lambda + \mu = 1\}$$

is contained in \mathcal{U} by checking on each coefficient separately.

Thus, the basis of open neighbourhoods of zero described by the sets of the form (16) consists of convex sets, and hence the higher topology on F is locally convex. \square

As we have done in the rest of cases, we may now describe the higher topology in terms of seminorms.

Proposition 9.3. *Let $k \in \mathbb{Z}$. Given a sequence $\rho_i \in \mathbb{Q}_{>0} \cup \{\infty\}$ for every $i \leq k$, such that $\rho_k < \infty$, consider the seminorm*

$$\|\cdot\| : \mathbb{K}((t)) \rightarrow \mathbb{R}, \quad \sum_{i \geq i_0} x_i t^i \mapsto \max_{i \leq k} \frac{|x_i|}{\rho_i}, \quad (17)$$

having in mind the convention that $a/\infty = 0$ for every $a \in \mathbb{R}_{\geq 0}$. The higher topology on F is defined by the set of seminorms specified by (17).

Proof. We will show that the seminorm $\|\cdot\|$ defined by (17) is the gauge seminorm attached to the basic open neighbourhood of zero \mathcal{U} given by (16).

Let $x = \sum_{i \geq i_0} x_i t^i \in F$ and $\rho > 0$. If $k < i_0$, we may take $\rho = 0$ and deduce that $q(x) = 0$.

Otherwise, $x \in \rho \mathcal{U}$ if and only if $x_i \in \rho D_{\rho_i}$ for every $i_0 \leq i \leq k$.

From this, we may deduce that $x \in \rho \mathcal{U}$ if and only if

$$\frac{|x_i|}{\rho_i} < \rho \quad \text{for every } i_0 \leq i \leq k. \quad (18)$$

Finally, the infimum value of ρ satisfying (18) is precisely the maximum of the values $|x_i|/\rho_i$ for $i_0 \leq i \leq k$. \square

We have described the higher topology on $\mathbb{K}((t))$ in a fashion that matches what has been done in the previous sections. However, this locally convex space often arises in functional analysis in the following way. We write

$$\mathbb{K}((t)) = \cup_{i \in \mathbb{N}} t^{-i} \cdot \mathbb{K}[[t]],$$

Each component in the union is isomorphic to $\mathbb{K}^{\mathbb{N}}$, topologized using the product topology, and the limit acquires the strict inductive limit locally convex topology.

It is known that $\mathbb{K}[[t]]$ is a Fréchet space, that is, complete and metrizable. As such, the two-dimensional local field $\mathbb{K}((t))$ is an LF-space and many of its properties may be deduced from the general theory of LF-spaces, see for example [8, §19]. In particular, $\mathbb{K}((t))$ is complete, bornological and nuclear.

10 A note on the characteristic p case

Let $k = \mathbb{F}_q$ be a finite field of characteristic p . In this section we will consider the two-dimensional local field $F = k((u))((t))$. It is a vector space both over the finite field k and over the local field $k((u))$.

The higher topology on F may be dealt with in two ways from a linear point of view. The first approach was started by Beilinson [1], and it regards F as a k -vector space. In this approach, k is regarded as a discrete topological field and the tools used are those of linear topology, see [6, §1] for an account.

The work developed in the previous sections of this work may be applied and we may regard F a locally convex $k((u))$ -vector space. In this section we will explain that in this case we have obtained nothing new.

A topology on a k -vector space is said to be linear if the filter of neighbourhoods of zero admits a collection of linear subspaces as a basis. A linearly topological vector space V is said to be linearly compact if any family $A_i \subset V$, $i \in I$ of closed affine subspaces such that $\bigcap_{i \in J} A_i \neq \emptyset$ for any finite set $J \subset I$, then $\bigcap_{i \in I} A_i \neq \emptyset$. Finally, a linearly topological vector space is locally compact if it has a basis of neighbourhoods of zero formed by linearly compact open subspaces.

Let \mathbf{Vect}_k be the category of linearly topological k -vector spaces. Similarly, let $\mathbf{Vect}_{k((u))}$ be the category of locally convex $k((u))$ -vector spaces.

Proposition 10.1. *The rule*

$$\mathbf{Vect}_{k((u))} \rightarrow \mathbf{Vect}_k, \quad (19)$$

which restricts scalars on $k((u))$ -vector spaces along the inclusion $k \hookrightarrow k((u))$ and preserves topologies and linear maps, is a functor which gives linearly compact topological k -vector spaces when applied to c -compact K -vector spaces.

Proof. Let V be a locally convex $k((u))$ -vector space, and let Λ denote an open lattice. As the lattice Λ is an $\mathcal{O}_{k((u))}$ -module and we have the inclusion $k \hookrightarrow \mathcal{O}_{k((u))} = k[[u]]$, it is also a k -vector space by restriction of scalars.

As the collection of open lattices Λ is a basis for the filter of neighbourhoods of zero, V is a linearly topological k -vector space and the first part of the proposition follows.

Finally, the definitions of c -compactness and linear compactness agree if in Proposition 1.7 we translate the words *closed convex subspace* by *closed affine subspace*. \square

What this result means from the point of view of two-dimensional local fields of positive characteristic is that the distinction between k -linear topology and $k((u))$ -locally convex topology is a matter of language.

The lack of an embedding of a finite field into a characteristic zero two-dimensional local field makes the linear topological approach unavailable in that setting; the locally convex approach to these fields is therefore to be regarded as analogous to the linear approach in positive characteristic. Similarly, the language of locally convex spaces is to be regarded as one which unifies the approach to the zero characteristic and positive characteristic cases.

11 Future work

We outline some directions which we consider interesting to explore in order to apply and extend the results in this work.

\mathcal{O} -linear locally convex approach to higher topology. In this work we have been able to deduce many properties about $K((t))$, either in an explicit or implicit way, from the fact that it is an LF-space, i.e.: an inductive limit of Fréchet spaces. This is not the case in mixed characteristics: the field $K\{\{t\}\}$ is not a direct limit of nice K -vector spaces. It is, however, a direct limit of \mathcal{O} -modules by construction.

The development of a theory of locally convex \mathcal{O} -modules, with topologies defined by seminorms, and the constructions arising within that theory, particularly those of initial and final locally convex topologies, would allow us to recover on one hand the results we have established for $K((t))$, and on the other hand they would let us describe $K\{\{t\}\}$ as a direct limit of perhaps *nice* \mathcal{O} -modules; this could be an extremely helpful contribution to the study of mixed characteristic two-dimensional local fields.

Generalization to higher local fields. If F is a characteristic zero n -dimensional local field, then it is possible to exhibit a field embedding $K \hookrightarrow F$ and treat F as a K -vector space. A higher topology on F may be constructed inductively using the same procedures outlined at the beginning of §3, see [9]. Therefore, it may be shown that these topologies define locally convex structures over K . Although the situation is slightly more complex, a systematic study of the functional theoretic properties of these locally convex spaces would be interesting to develop.

Study of $\mathcal{L}(F)$. As we have explained, the ring of continuous K -linear endomorphisms of a two-dimensional local field can be topologized and studied from a functional analytic point of view. It contains several relevant two-sided ideals defined by imposing certain finiteness conditions to endomorphisms. The most important of such ideals is the subspace of nuclear maps. Nuclear endomorphisms of a locally convex space play a distinguished role in the study of the properties of such space. In particular, the usual trace map on finite-rank operators extends by topological arguments to the subspace of nuclear endomorphisms. Establishing a characterization of nuclear endomorphisms of two-dimensional local fields is an affordable goal.

Multiplicative theory of two-dimensional local fields. Multiplication $\mu : F \times F \rightarrow F$ on a two-dimensional field F is not continuous as explained

in §3.3. It is a well-known fact that the map μ is sequentially continuous, and the sequential topological properties of higher topologies have been studied and applied successfully to higher class field theory [4] and to topologize sets of rational points of schemes over higher local fields [2].

However, for any $x \in F$, the linear maps

$$\begin{aligned}\mu(x, \cdot) : F &\rightarrow F, \\ \mu(\cdot, x) : F &\rightarrow F\end{aligned}$$

are continuous. This means that, in the terms of [13, §17], μ is a separately continuous bilinear map and therefore induces a continuous linear map of locally convex spaces

$$\mu : F \otimes_{K,\iota} F \rightarrow F, \quad (20)$$

where $F \otimes_{K,\iota} F$ stands for the tensor product $F \otimes_K F$ topologized using the inductive tensor product topology.

This suggests that besides the applications of the theory of semitopological rings to the study of arithmetic properties of higher local fields [15], we have the following new approach to the topic: a two-dimensional local field F is a locally convex K -vector space endowed with a continuous linear map $\mu : F \otimes_{K,\iota} F \rightarrow F$ satisfying the usual axioms of multiplication.

After Proposition 4.8 and Corollary 5.12, another possible way to look at a two-dimensional local field and deal with its multiplicative structure is as a bornological K -algebra, that is: F is a K -algebra endowed with a bornology (that generated by bounded submodules or compactoid submodules) such that all K -algebra operations

$$\begin{aligned}\sigma : F \times F &\rightarrow F \quad (\text{addition}), \\ \varepsilon : K \times F &\rightarrow F \quad (\text{scalar multiplication}), \\ \mu : F \times F &\rightarrow F\end{aligned}$$

are bounded.

It is interesting to decide if the arithmetic properties of F can be recovered from these contexts, and it would even more interesting to establish new connections between this functional analytic approach to higher topology and the arithmetic of F .

Functional analysis on adelic rings and modules over them. There are several two-dimensional adelic objects which admit a formulation as a restricted product of two-dimensional local fields and their rings of integers, which in our characteristic zero context were introduced by Beilinson and Fesenko [10, §8]. From what we have exhibited in this work, at least in dimension two, these adelic objects may be studied using the theory of locally convex spaces, archimedean or nonarchimedean.

Topological approach to higher measure and integration. The study of measure theory, integration and harmonic analysis on two-dimensional local fields is an interesting problem. A theory of measure and integration has been developed on two-dimensional local fields F by lifting the Haar measure of the local field \overline{F} [3], [12]. This theory relies heavily on the relation between F and \overline{F} .

The approach to measure and integration on F using the functional theoretic tools arising from the relation between F and K could yield an alternative integration theory.

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